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ENTROPY BOUNDS FOR PERFECT MATCHINGS AND HAMILTONIAN CYCLES*

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For a graph $G=(V,E)$ and $\mathbf{x}\colon E\to\Re^+$ satisfying $\sum_{e\ni v}\mathbf{x}_e=1$ for each $v\in V$, set $h(\mathbf{x})=\sum_e \mathbf{x}_e\log(1/\mathbf{x}_e)$ (with $\log=\log_2$). We show that for any n-vertex G, random (not necessarily uniform) perfect matching $\mathbf{f}$ satisfying a mild technical condition, and $\mathbf{x}_e=\Pr(e\in\mathbf{f})$,

$$H(\mathbf{f})<h(\mathbf{x})-\frac{n}{2}\log e+o(n)$$

(where H is binary entropy). This implies a similar bound for random Hamiltonian cycles.

Specializing these bounds completes a proof, begun in [6], of a quite precise determination of the numbers of perfect matchings and Hamiltonian cycles in Dirac graphs (graphs with minimum degree at least $n/2$) in terms of $h(G):=\max\sum_e \mathbf{x}_e\log(1/\mathbf{x}_e)$ (the maximum over $\mathbf{x}$ as above). For instance, for the number, $\Psi(G)$, of Hamiltonian cycles in such a G, we have

$$\Psi(G)=\exp_2[2h(G)-n\log e-o(n)].$$

1. Introduction

Our main result is Theorem 1.2. We begin by recalling a statement from [6], completion of whose proof was the original motivation for the present work.

For a graph $G=(V,E)$ and edge weighting $\mathbf{x}\colon E\to\Re^+$, set $h(\mathbf{x})=\sum_e \mathbf{x}_e\log(1/\mathbf{x}_e)$ (where $\log=\log_2$). We will call this the *entropy* of $\mathbf{x}$ (but note it is not really entropy since $\sum\mathbf{x}_e$ will not usually be 1). Call an edge weighting $\mathbf{x}$ *proper* if $\sum_{e\ni v}\mathbf{x}_e=1$ for each $v\in V$. (Such an $\mathbf{x}$ is often called a "perfect fractional matching".) Finally, let $h(G)$ (the "entropy" of G) be the

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maximum of $h(\mathbf{x})$ over proper edge weightings $\mathbf{x}$. (In the absence of proper weightings we may set $h(G)=0$, but this won't be an issue here.)

We use $\Phi(G)$ and $\Psi(G)$ for the numbers of perfect matchings and Hamiltonian cycles of a graph G. Recall that a graph is *Dirac* if it has minimum degree at least $n/2$. (The reference is of course to [7], which proves that such graphs are Hamiltonian.)

Theorem 1.1. *For any n-vertex Dirac graph G,*

$$\log \Psi(G) = 2h(G) - n\log e - o(n), \tag{1}$$

(note $o(n)$ is not necessarily positive), and, if n is even,

$$\log \Phi(G) = h(G) - (n/2)\log e - o(n). \tag{2}$$

The lower bounds (that is, that the l.h.s.'s of (1) and (2) are larger than the r.h.s.'s) were proved in [6]. Of course the second follows from the first, since, trivially, $2\Psi(G)\le\Phi^2(G)$ for any G with an even number of vertices. One perhaps surprising consequence of Theorem 1.1 is that for Dirac graphs this inequality is close to the truth, namely

$$\Psi(G) = e^{-o(n)}\Phi^2(G).$$

(Throughout our discussion n is the default for $|V(G)|$.)

When combined with a lower bound on $h(G)$ proved in [6], the lower bound in (1) says in particular that for any (n-vertex) Dirac G,

$$\Psi(G) \ge n!/(2+o(1))^n\,; \tag{3}$$

this proved a conjecture from [12] which was the starting point for [6] and the present work.

The upper bounds in Theorem 1.1 are essentially instances of the next result. We regard a perfect matching as either an involution $f\colon V\to V$ (in the obvious way), or as a set of edges.

Theorem 1.2. *For any n-vertex graph G:*
(a) if $\mathbf{f}$ is a random (not necessarily uniform) perfect matching of G and $\mathbf{x}_e=\Pr(e\in\mathbf{f})$ for $e\in E$, then

$$H(\mathbf{f}) < h(\mathbf{x}) - \frac{n}{2}\log e + o(n) \tag{4}$$

provided

$$\sum \mathbf{x}_e^2\log(2/\mathbf{x}_e) < o(n); \tag{5}$$

(b) if $\mathbf{C}$ *is a random Hamiltonian cycle of* G *and* $\mathbf{x}_e = \frac{1}{2}\Pr(e \in \mathbf{C})$ *satisfies* (5), *then*

$$H(\mathbf{C}) < 2h(\mathbf{x}) - n\log e + o(n). \tag{6}$$

Recall that Brégman's Theorem [2] says (in graph-theoretic language) that for a bipartite graph G on vertex set $X \cup Y$, one has

$$\log \Phi(G) \le \sum_{x \in X} \frac{1}{d_x} \log d_x! \tag{7}$$

(where d_x is the degree of x). This is sharp for any disjoint union of complete, balanced bipartite graphs. A nonbipartite version, which is sharp for the same examples, says[1]: for *any* G,

$$\log \Phi(G) \le \sum_{v \in V(G)} \frac{1}{2d_v} \log d_v!\,. \tag{8}$$

The l.h.s. of (7) is trivially at most $\sum \log d_x$, while that of (8) is at most $\frac{1}{2}\sum \log d_v$ – an application of *Shearer's Entropy Lemma* ([3, p. 33], but see e.g. [9, Lemma 2.1] for the relevant form) to $\log \Phi(G) = H(\mathbf{f})$, with $\mathbf{f}$ a uniform (random) perfect matching (and H binary entropy; for entropy basics see [10] or [4]). Thus for large degrees, each of (7), (8) gains about $\log e$ per vertex.

Theorem 1.2 does something similar in a more general setting, but with a small loss of precision. Though the error makes no difference for Theorem 1.1, it would be nice to have versions of Theorem 1.2 that include (7) and (8). Let $\varphi(x) = 2^{-x}\log \Gamma(2^x + 1)$ (so $\varphi(\log d) = d^{-1}\log d!$ for positive integer d). Could it be that for a general (i.e., not necessarily uniform) random perfect matching $\mathbf{f}$, (7) and (8) extend to

$$H(\mathbf{f}) \le \sum_{x \in X} \varphi(H(\mathbf{f}(x)))$$

and

$$H(\mathbf{f}) \le \frac{1}{2}\sum_{v} \varphi(H(\mathbf{f}(v)))?$$

[1] This was proved, but not published, a few years ago by L. Lovász and the second author, and has since appeared in [8] and [1].

2. Proof of Theorem 1.1

To see that Theorem 1.2 implies the upper bounds in Theorem 1.1, we need to show that (5) holds when G is Dirac and $\mathbf{f}$ (resp. $\mathbf{C}$) is a uniform perfect matching (resp. Hamiltonian cycle). (Note it is not necessarily true here that *all* $\mathbf{x}_e$'s are small. Suppose for instance that $V(G)$ is the disjoint union of A and B, each of size $n/2$, and $E(G)$ consists of a single edge e contained in A together with all edges not contained in A. Then for $\mathbf{f}$ a uniform perfect matching and $\mathbf{x}$ as above we have $\mathbf{x}_e = 1/3$.)

The verifications of (5) in these two cases (uniform $\mathbf{f}$, uniform $\mathbf{C}$) are essentially identical, so we just consider the first. Note that (5) is equivalent to the statement that for some $\alpha = o(1)$ we have $\sum\{\mathbf{x}_e \colon \mathbf{x}_e \geq \alpha\} < o(n)$. (This obviously implies (5). For the converse, let $f(\alpha) = \sum\{\mathbf{x}_e \colon \mathbf{x}_e \geq \alpha\}$ and $\alpha_0 = \max\{\alpha \colon f(\alpha) \geq \alpha n\}$. Then (i) for $\alpha = \alpha_0$, $o(n) > \sum \mathbf{x}_e^2 \geq (f(\alpha)/\alpha)\alpha^2 = \alpha f(\alpha) \geq \alpha^2 n$ implies $\alpha_0 < o(1)$, and (ii) for any α with $\alpha_0 < \alpha < o(1)$, we have $f(\alpha) < \alpha n < o(n)$.)

Assume then that $\alpha = \alpha(n)$ is both $\omega(1/\sqrt{\log n})$ and $o(1)$, and suppose for a contradiction that $\mathcal{B}$ is a set of αn edges, each of weight at least α, and let $\sum_{e \in \mathcal{B}} \mathbf{x}_e = \gamma n$; thus $\alpha^2 \leq \gamma \leq \alpha$. Then Shearer's Lemma and concavity of $h(z) := z \log(1/z)$ give

$$
\begin{aligned}
H(\mathbf{f}) &\leq \frac{1}{2} \sum_{v \in V} H(\mathbf{f}(v)) = h(\mathbf{x}) \\
&= \sum_{e \in \mathcal{B}} h(\mathbf{x}_e) + \sum_{e \notin \mathcal{B}} h(\mathbf{x}_e) \\
&\leq \alpha n h\left(\frac{\gamma n}{\alpha n}\right) + \left(\binom{n}{2} - \alpha n\right) h\left(\frac{n/2 - \gamma n}{\binom{n}{2} - \alpha n}\right) \\
&< \alpha n + (1/2 - \gamma) n (\log n - \log(1 - 2\gamma)) \\
&< (1/2 - \gamma) n \log n + 3\alpha n = (n/2)(\log n - \omega(1)).
\end{aligned}
$$

But we know from (3) that $H(\mathbf{f}) > (n/2)(\log(n/2) - \log e - o(1))$, and this gives the desired contradiction. ∎

3. Proof of Theorem 1.2

Observe, first of all, that (a) easily implies (b). To see this for even n, let $\mathbf{f}$ be the random matching gotten by first choosing $\mathbf{C}$ and then choosing each of the two perfect matchings contained in $\mathbf{C}$ to be $\mathbf{f}$ with probability $1/2$.

Then the $\mathbf{x}$ in (a) is the same as that in (b). On the other hand, since the laws of $\mathbf{f}$ and $\mathbf{C}\setminus\mathbf{f}$ are the same, we have

$$H(\mathbf{C})+1=H(\mathbf{f},\mathbf{C}\setminus\mathbf{f})\le 2H(\mathbf{f}),$$

so that (6) follows from (4).

When n is odd, we have a simple reduction to the even case: Let G' be gotten from G by adding a new vertex z adjacent to all vertices of G, and let $\mathbf{C}'$ be the random Hamiltonian cycle of G' gotten by choosing an edge xy uniformly from $\mathbf{C}$ and replacing it by the path (x,z,y). Then for $\mathbf{x}'_e=\frac{1}{2}\Pr(e\in\mathbf{C}')$ we have

$$\mathbf{x}'_e=\begin{cases}(1-1/n)\mathbf{x}_e & \text{if } e\in E(G)\\ 1/n & \text{if } z\in e,\end{cases}$$

whence

$$h(\mathbf{x}')=\sum_{e\in E(G)}\left(1-\frac{1}{n}\right)\mathbf{x}_e\log\frac{n}{(n-1)\mathbf{x}_e}+n\cdot\frac{1}{n}\log n<h(\mathbf{x})+\log n+1.$$

On the other hand (noting $\mathbf{x}'$ clearly satisfies (5)),

$$H(\mathbf{C})=H(\mathbf{C}')-\log n<2h(\mathbf{x}')-(n+1)\log e+o(n)-\log n,$$

and combining the last two observations again gives (6).

We now turn to (a). Our approach here is similar to Radhakrishnan's in [11] (which is in turn similar to Schrijver's [13], but the entropy viewpoint of [11] is surely the correct one). A key difference is that we cannot afford to bound the summand in (9) by the log of the number of possibilities for $\mathbf{f}(v)$ given $\mathbf{Y}_v$, as one would at a corresponding point in [11].

Note that we may assume $G=K_n$ (since we may simply regard $\mathbf{f}$ as a random matching of K_n that makes no use of edges not belonging to G). Strangely, this trivial assumption yields a significant simplification of the original proof of (a) given in [5].

We use f_B for the restriction of f to $B\subseteq V$. For a permutation σ of V and $v\in V$, set $B(\sigma,v)=\{w\in V:\sigma(w)<\sigma(v)\}$. Let $\mathbf{s}$ be a random (uniform) permutation of V and $\mathbf{Y}_v=(\mathbf{s},\mathbf{f}_{B(\mathbf{s},v)})$. Then (by the "chain rule")

$$\begin{aligned}H(\mathbf{f})&=\frac{1}{n!}\sum_\sigma\sum_v H\big(\mathbf{f}(v)\mid\mathbf{f}_{B(\sigma,v)}\big)\\&=\sum_v\sum_\sigma\sum_g\frac{1}{n!}\Pr\big(\mathbf{f}_{B(\sigma,v)}=g\big)H(\mathbf{f}(v)\mid\sigma,g)\\&=\sum_v H(\mathbf{f}(v)\mid\mathbf{Y}_v),\end{aligned}\tag{9}$$

where σ ranges over possible values of $\mathbf{s}$ and, given σ, g ranges over possible values of $\mathbf{f}_{B(\sigma,v)}$ (and the conditioning on (σ,g) has the obvious meaning).

Now let

$$\mathbf{Z}_v = \begin{cases} (\{\mathbf{f}(v)\}, \Lambda) & \text{if } \mathbf{s}(\mathbf{f}(v)) < \mathbf{s}(v) \\ V \setminus \{v\} \setminus (B(\mathbf{s},v) \cup \mathbf{f}(B(\mathbf{s},v))) & \text{otherwise.} \end{cases}$$

This is basically the set of possibilities remaining when we come to specify $\mathbf{f}(v)$ (following the ordering $\mathbf{s}$), but it will be helpful to distinguish the two ways in which this set can be a singleton $\{y\}$ (namely we can have $\mathbf{f}(v)=y$ with $\mathbf{s}(y)<\mathbf{s}(v)$ or with $\mathbf{s}(y)>\mathbf{s}(v)$ and $V\setminus\{v\}\setminus(B(\mathbf{s},v)\cup\mathbf{f}(B(\mathbf{s},v)))=\{y\}$).

Since $\mathbf{Z}_v$ is a function of $\mathbf{Y}_v$, we have $H(\mathbf{f}(v)\,|\,\mathbf{Y}_v)\le H(\mathbf{f}(v)\,|\,\mathbf{Z}_v)$ and so (by (9))

$$H(\mathbf{f}) \le \sum_v H(\mathbf{f}(v) \mid \mathbf{Z}_v). \tag{10}$$

Our main task is bounding $H(\mathbf{f}(v)\,|\,\mathbf{Z}_v)$.

From this point we fix v and write $\mathbf{Z}$ in place of $\mathbf{Z}_v$. We use y for values of $\mathbf{f}(v)$ and Z for values of $\mathbf{Z}$ *not of the form* $(\{\mathbf{f}(v)\},\Lambda)$. We use p_y for $\Pr(\mathbf{f}(v)=y)$, but apart from this let $\Pr(Z)=\Pr(\mathbf{Z}=Z)$, $\Pr(Z\,|\,y)=\Pr(\mathbf{Z}=Z\,|\,\mathbf{f}(v)=y)$ and so on.

Since $H(\mathbf{f}(v)\,|\,\mathbf{Z}=(y,\Lambda))=0$, we have

$$\begin{aligned} H(\mathbf{f}(v) \mid \mathbf{Z}) &= \sum_Z \Pr(Z) \sum_y \Pr(y \mid Z) \log \frac{1}{\Pr(y \mid Z)} \\ &= \sum_y \sum_Z \Pr(y,Z) \log \frac{\Pr(Z)}{\Pr(y,Z)} \\ &= \sum_y p_y \left[\frac{1}{2} \log \frac{1}{p_y} + \sum_Z \Pr(Z \mid y) \log \frac{\Pr(Z)}{\Pr(Z \mid y)} \right] && (11) \\ &= \frac{1}{2} H(\mathbf{f}(v)) + \sum_y p_y \sum_Z \Pr(Z \mid y) \log \frac{\Pr(Z)}{\Pr(Z \mid y)}. && (12) \end{aligned}$$

(For (11) notice that, for any y, $\sum \Pr(Z\,|\,y)=1/2$, since $\mathbf{f}(v)$ and the event $\{\mathbf{s}(\mathbf{f}(v)) < \mathbf{s}(v)\}$ are independent.) We want to show that the second term in (12) is at most about $(-\log e)/2$. (Note $\sum H(\mathbf{f}(v))=2h(\mathbf{x})$.)

Fix $y\in V\setminus\{v\}$ with $p_y>0$. Let Q be the event $\{\mathbf{s}(\mathbf{f}(v))>\mathbf{s}(v)\}$, and for $k\in\{1,\dots,n-1\}$, set

$$q_k = \sum\{\Pr(Z \mid y)\colon Z \ni y, |Z| = k\} = \Pr(Q, |\mathbf{Z}| = k \mid \mathbf{f}(v) = y),$$

$$r_k = \sum\{\Pr(Z)\colon Z \ni y, |Z| = k\} = \Pr(Q, |\mathbf{Z}| = k, \mathbf{Z} \ni y).$$

(Notice that "$|\mathbf{Z}|=k$" and "$\mathbf{Z}\ni y$" make sense once we know Q holds, and that it is not really necessary to specify "$Z\ni y$" in the definition of q_k.) We have

$$\sum_Z \Pr(Z\mid y)\log\frac{\Pr(Z)}{\Pr(Z\mid y)}$$

$$= \sum_k q_k \sum\left\{\frac{\Pr(Z\mid y)}{q_k}\log\frac{\Pr(Z)}{\Pr(Z\mid y)} : Z\ni y, |Z|=k\right\}$$

$$\le \sum_k q_k \log\frac{r_k}{q_k}, \tag{13}$$

so that (noting $p_y=\mathbf{x}_{vy}$) Theorem 1.2 will follow from

$$\sum_k q_k \log\frac{r_k}{q_k} < -\frac{1}{2}\log e + O(p_y\log(2/p_y)+n^{-1}\log n). \tag{14}$$

The values of q_k and r_k are as follows. Let

$$s_k = \Pr(Q, |\mathbf{Z}|=k, \mathbf{Z}\ni y \mid \mathbf{f}(v)\neq y).$$

Then $r_k=p_yq_k+(1-p_y)s_k$ and (with justifications to follow) we have

$$q_k = \begin{cases} 1/n & \text{if } k=1,3,\ldots,n-1 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

and

$$s_k = \begin{cases} \frac{k-1}{n(n-2)} & \text{if } k=1,3,\ldots,n-1 \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

In fact these identities hold even if we condition on the value of $\mathbf{f}$; that is, (15) is still correct if we replace q_k by

$$q_k(f) = \Pr(Q, |\mathbf{Z}|=k \mid \mathbf{f}=f)$$

for any f (but we only use this when $f(v)=y$), and (16) is correct with s_k replaced by

$$s_k(f) = \Pr(Q, |\mathbf{Z}|=k, \mathbf{Z}\ni y \mid \mathbf{f}=f)$$

provided $f(v)\neq y$. (Given $\{\mathbf{f}=f\}$ and Q we may order the edges of f by letting e precede e' if the first vertex of e (under the random permutation $\mathbf{s}$) precedes that of e'. Then $|\mathbf{Z}|-1$ is twice the number of edges of f that follow the edge containing v in this order, and this number is uniform from $\{0,\ldots,n/2-1\}$; and then, given $|\mathbf{Z}|$, each vertex other than v and $f(v)$ – in particular y if $f(v)\neq y$ – is equally likely to belong to $\mathbf{Z}$.)

Thus,

$$\sum_k q_k \log \frac{r_k}{q_k} = \frac{1}{n}\sum_{t=1}^{n/2} \log\left[p_y + (1-p_y)\frac{2t-2}{n-2}\right]$$
$$\leq \frac{1}{2n}\sum_{l=2}^{n} \log[p_y + (1-p_y)l/n]$$
$$= \frac{1}{2n}\sum_{l=2}^{n} \log(l/n) + \frac{1}{2n}\sum_{l=2}^{n} \log(1+p_y(n/l-1))$$
$$< -\frac{1}{2}\log e + O\big(n^{-1}\log n\big) + \frac{1}{2n}\sum_{l=2}^{n}\log(1+p_y n/l). \tag{17}$$

For the last sum we have

$$\sum_{l=2}^{n}\log(1+p_y n/l) < \sum_{2\leq l\leq p_y n}\log(2p_y n/l) + \log e \sum_{l>p_y n} p_y n/l$$
$$< p_y n\log(2e) + p_y n \log(1/p_y),$$

so that the expression in (17) is less than

$$-\frac{1}{2}\log e + O\big(p_y\log(2/p_y) + n^{-1}\log n\big),$$

and we have (14). ∎

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